Pergamon

Int. J. Solids Structures Vol. 34, No. 4, pp. 415–434, 1997

0020–7683/97 $17.00 + .00

PII: S0020–7683(96)00023–6

PARAMETRIZATION IN LAMINATE DESIGN FOR OPTIMAL COMPLIANCE

V. B. HAMMER*, M. P. BENDSØE†, R. LIPTON‡ and P. PEDERSEN*

* Department of Solid Mechanics, Technical University of Denmark, DK-2800 Lyngby, Denmark

† Mathematical Institute, Technical University of Denmark, DK-2800 Lyngby, Denmark

‡ Department of Mathematical Sciences, Worcester Polytechnic Institute, Worcester, MA 01609, U.S.A.

(*Received* 21 *June* 1995; *in revised form* 12 *February* 1996)

Abstract—In this paper we consider the maximal stiffness design of laminated plates subjected to single and multiple loads. The stiffness of the laminates are parametrized in terms of the so-called lamination parameters. These express the relation between the material parameters for the laminate and the laminate lay-up and are given as moments of the trigonometric functions that appear in the well-known rotation formulae for stiffness matrices. These relations are here given in a form suitable for optimization studies. The conditions for the laminate itself to be orthotropic are also given directly in terms of the lamination parameters.

The design problem is analyzed by performing a reformulation to an equivalent problem which is local in character and it is shown how this, together with an enlargement of the design space to allow for out of plane chattering designs, leads to a significant simplification of the problem. Thus, the number of variables is reduced to only four for the stiffness problem at hand, even in the general case with coupling stiffnesses and multiple loads. Moreover, in the special case of in-plane loads, the optimal solution for each design element of the plate can be realized as a single rotated ply of material or in special strain situations by two plies. A computational solution procedure for the simplified problem is described and several numerical examples illustrate basic features of the design approach. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

Problem formulation and model parametrization plays a significant role in the success of applying optimization techniques to the design of structures. A proper choice of design parameters enables one to perform analytical studies such as sensitivity analysis and problem reduction as well as for the development of efficient computational solution procedures.

The present study is concerned with the optimal design of the lay-up of laminated plates for maximum stiffness. We consider optimization with respect to the ply thicknesses, fiber orientations and the stacking sequence of the laminates, keeping the ply material properties and the shape of the plate fixed. Instead of working directly with these design parameters we use the so-called lamination parameters for the design parametrization. The lamination parameters, first introduced by Miki (1982), represent the effective, integrated properties of the laminate and are given as moments relative to the plate mid-plane of the trigonometric functions entering in the frame rotation formulas for stiffness matrices. In this way the properties related to the stiffness of the laminates are emphasized in the optimization model, while the realization of the optimal effective properties is postponed for subsequent post-processing. Introduction of the lamination parameters also means that the number of design variables is reduced to twelve, independently of the number of plies and stacking sequence. Moreover, the stiffness of the plate depends linearly on the lamination parameters which constitute a convex set, thus endowing the optimization problems with nice properties.

The most direct approach to laminate design is to express the optimization problems directly in terms of the stacking sequence and the fiber orientations and thicknesses of the individual plies. Ultimately, the optimization problem involves a mix of integer variables (number of plies) and real variables (orientations and thicknesses). Much effort has been devoted to solving laminate design problems in this form, typically by combining methods for differentiable optimization with a variety of methods for integer programming. We will not survey such work here, but refer to the vast literature (see, e.g., chapter 11 in Haftka *et al.* (1990) and Fang and Springer (1993), for references to the literature).

The design parametrization through lamination parameters has been used in laminate design in many different problems where the overall properties of the laminate are the governing feature, such as in stiffness optimization, Miki (1982), Fukunaga and Sekine (1993), in vibration optimization, Fukunaga *et al.* (1994), Grenestedt and Gudmundson (1993), and in buckling optimization, Grenestedt (1991), Miki and Sugiyama (1993), and Fukunaga and Sekine (1994). A key point in the application of the lamination parameters for design purposes is the identification of the range of the lamination parameters as this information should be included in the problem formulation as design constraints in order to assure that the resulting designs can be realized by physical lay-ups. There are twelve lamination parameters in all, corresponding to zero order (for membrane), first order (for extension-bending coupling) and second order (for bending) moments for four trigonometric functions for the variation of ply rotation angle through the plate thickness. The lamination parameters can thus not range over the full unit cube $[-1, 1]^{12}$ in $\mathbb{R}^{12}$. For the four lamination parameters characterizing the pure membrane or the pure bending case the design domain has been determined conclusively (Fukunaga and Sekine (1994); see also comments below), but an analytical characterization of the design domain for all twelve parameters is, to the best of our knowledge, still to be determined. Thus, in general, the literature is concerned with situations where a limitation of the number of design variables included in the optimization is achieved by constraining the laminates to be orthotropic, Fukunaga and Vanderplaats (1991), Grenestedt (1991), and Miki and Sugiyama (1993), or by avoiding the extension-bending coupling effects through restricting the laminates to be symmetric, Miki (1982), Fukunaga *et al.* (1994) and Fukunaga and Sekine (1994). The reduction of the design freedom to a few laminate types in these studies have also been necessary for identifying lay-ups with prescribed lamination parameters, as the general problem is unsolved except for very special combinations of the lamination parameters or for the symmetric eight ply case, Fukunaga and Sekine (1992).

The present study is strongly inspired by recent work on free material design and the homogenization method for topology optimization. This encompasses both the approach to the formulation and analysis of the problem as well as to the design parametrization. For the former this implies that the optimization problem is made local in character, allowing for an analytical derivation of the optimal local properties of material, Bendsøe *et al.* (1995), Jog *et al.* (1994), Lipton (1994a), Díaz *et al.* (1994), Allaire and Kohn (1993), Cherkaev and Palais (1995). As to the latter, we choose to extend the design space instead of reducing the feasible designs to, say, only symmetric laminates. Thus the design space is enlarged to include out of plane chattering designs, thereby allowing infinitely many small variations of the fiber orientation in each point through the thickness for each design domain of the plate. This was also used by Grenestedt and Gudmundson (1993), in their proof of the convexity of the feasibility domain of the lamination parameters. Also, it corresponds closely to the introduction of in plane chattering designs in the form of periodic composites for topology design and of rib-reinforced plates in plate design, see for example the literature surveys in the monographs Bendsøe (1995), and Lurie (1993). In the present situation the out of plane chattering designs result in a linear relation between stiffness and design variables (as noted above), assuring existence of solutions without in plane chattering and facilitating analysis in a way analogous to the simplification achieved when considering the free material design (there the full stiffness tensor is considered as the design variables).

Moments of trigonometric functions also appear naturally in the homogenization method for topology optimization. Here the so-called finite rank layered materials consisting of combinations of layers of flexible (void) and stiff (solid) material at various scales provide the optimal microstructures required for stiffness design and it is possible to express the effective material properties of such materials (in dimension two) in terms of trigonometric moments, Avellaneda and Milton (1989). The stiffness relation is in this case non-linear, but concave. Such a parametrization has been used in topology optimization as well as for optimization of plate reinforcement (see, for example, Díaz *et al.* (1994), Lipton (1994b)), and the problem reduction and computational scheme presented here is strongly related to these works. A main feature of the study of finite rank layered materials for design is also in that case the identification of the set of admissible moments as well as the realization of specific moments from layered materials with a finite (low) number of layers. It turns out that, for any given set of moments, a composite with at most three layers can be constructed analytically, see Lipton (1994b). This approach is used here to show that lay-ups of at most three plies serve to generate all extended laminate parameters associated with the membrane stiffness matrix. More generally, one sees that the Young's measure associated with a chattering sequence of designs is given in terms of a probability density with support on at most three different ply angles, see Appendix.

The design formulation in terms of the extended lamination parameters together with the local optimization of the lamination parameters outlined above leads to significant simplifications of the stiffness optimization problem. Thus the localization holds both through the thickness as well as in plane, implying that only the range of the four zero order moments is needed for solving the problem. For pure membrane problems the local optimization of the laminate yields an optimal solution in the form of a non-chattering design (in plane and out of plane), and this holds for single as well as multiple load designs. One can also conclude that an optimal design can be realized with at most two plies (per design area), and this is also seen in the numerical implementation. Moreover, for a single load scenario the realization can consist of just a single ply or a cross-ply laminate, depending on the local strain situation.

For a complete derivation of the optimal laminate the local results must be combined with a computation of the displacements (the strains and curvatures) of the plate under the given loading cases. This leads naturally to a computational scheme which iterates between finite element based displacement analysis for fixed lamination parameters and the local optimization of the lamination parameters for fixed strains. The solution to the latter problem generates a displacement problem (in the form of a minimum potential energy formulation) which is convex but non-smooth. The non-smoothness can be circumvented by adding a penalty in the lamination parameters (a viscosity approach) and the numerical experiments show that with this penalty the use of a mathematical programming method for solving the local problem combined with a standard linear displacement analysis results in a very reliable iterative computational scheme.

2. PARAMETRIZATION BY LAMINATION PARAMETERS

In the following the constitutive relations for a single ply of material and for a laminate of several plies are stated. We will not limit the presentation to any specific material type or laminate type, but treat the general case of anisotropic plies in any lay-up. The relations can easily be simplified to, say the case of a symmetric laminate built up of orthotropic layers.

The elasticity tensor $\mathbf{C}_{ijkl}$ will, for convenience, be written as a matrix as

$$[\mathbf{C}]_X = \begin{bmatrix} C_{1111} & C_{1122} & \sqrt{2}C_{1112} \\ C_{1122} & C_{2222} & \sqrt{2}C_{2212} \\ \sqrt{2}C_{1112} & \sqrt{2}C_{2212} & 2C_{1212} \end{bmatrix}_X . \qquad (1)$$

The index indicates that the constitutive parameters are given in the coordinate system X. In another coordinate system x rotated the angle γ positive anti-clockwise relative to the X-system, $[\mathbf{C}]_x$ is most easily expressed using the material parameters $C_{1\text{-}7}$, introduced by Tsai and Hahn (1980). To ease the formulation, later on, the constitutive matrix $[\mathbf{C}]_x$ is written in terms of five symmetric matrices containing the material parameters as:

$$[\mathbf{C}]_x = [\Upsilon_0] + [\Upsilon_1]\cos 2\gamma + [\Upsilon_2]\cos 4\gamma + [\Upsilon_3]\sin 2\gamma + [\Upsilon_4]\sin 4\gamma$$

$$[\Upsilon_0] = \begin{bmatrix} C_1 & C_4 & 0 \\ C_4 & C_1 & 0 \\ 0 & 0 & 2C_5 \end{bmatrix}, \quad [\Upsilon_1] = \begin{bmatrix} C_2 & 0 & \sqrt{2}C_6 \\ 0 & -C_2 & \sqrt{2}C_6 \\ \sqrt{2}C_6 & \sqrt{2}C_6 & 0 \end{bmatrix},$$

$$[\Upsilon_2] = \begin{bmatrix} C_3 & -C_3 & \sqrt{2}C_7 \\ -C_3 & C_3 & -\sqrt{2}C_7 \\ \sqrt{2}C_7 & -\sqrt{2}C_7 & -2C_3 \end{bmatrix}, \quad [\Upsilon_3] = \begin{bmatrix} 2C_6 & 0 & -\frac{1}{\sqrt{2}}C_2 \\ 0 & -2C_6 & -\frac{1}{\sqrt{2}}C_2 \\ -\frac{1}{\sqrt{2}}C_2 & -\frac{1}{\sqrt{2}}C_2 & 0 \end{bmatrix},$$

$$[\Upsilon_4] = \begin{bmatrix} C_7 & -C_7 & -\sqrt{2}C_3 \\ -C_7 & C_7 & \sqrt{2}C_3 \\ -\sqrt{2}C_3 & \sqrt{2}C_3 & -2C_7 \end{bmatrix} \tag{2}$$

where the material parameters C_{1-7} are expressed as

$$\begin{aligned} C_1 &= \tfrac{1}{2}(C_{1111} + C_{2222})_X - C_3 \\ C_2 &= \tfrac{1}{2}(C_{1111} - C_{2222})_X \\ C_3 &= \tfrac{1}{8}(C_{1111} + C_{2222} - 2C_{1122} - 4C_{1212})_X \\ C_4 &= (C_{1122})_X + C_3 \\ C_5 &= (C_{1212})_X + C_3 = \tfrac{1}{2}(C_1 - C_4) \\ C_6 &= \tfrac{1}{2}(C_{1112} + C_{2212})_X \\ C_7 &= \tfrac{1}{2}(C_{1112} - C_{2212})_X. \end{aligned} \tag{3}$$

If the material is orthotropic in the X-system, $C_6 = C_7 = 0$, and in the case of an isotropic material $C_2 = C_3 = 0$ as well. On the other hand, if the material is orthotropic in some reference system x, which may not be evident at a first glance on the [**C**]-matrix, the direction of orthotropy found by the condition $C_{1112} = C_{2212} = 0$ and is given by

$$\tan 2\gamma = -\frac{2C_6}{C_2}, \quad \tan 4\gamma = -\frac{C_7}{C_3}. \tag{4}$$

From trigonometric relations between $\tan 2\gamma$ and $\tan 4\gamma$ one obtains the following general expression, Pedersen (1990b):

$$C_7 C_2^2 - 4C_7 C_6^2 - 4C_6 C_3 C_2 = 0. \tag{5}$$

This equality provides necessary and sufficient conditions for the material to be orthotropic.

We consider a laminate of the thickness h made from several plies. Here the orientation of the ith ply is specified by γ_i and z_i gives the location (dimensionless) of the interface between ply i and $i+1$, see Fig. 1. All the plies consist of the same anisotropic material.

In the classical plate theory the global relation between the membrane forces and moments per unit length $\{\mathbf{N}\}$, $\{\mathbf{M}\}$ and the mid-plane strains $\{\boldsymbol{\varepsilon}^0\}$ and curvatures $\{\boldsymbol{\kappa}\}$ is

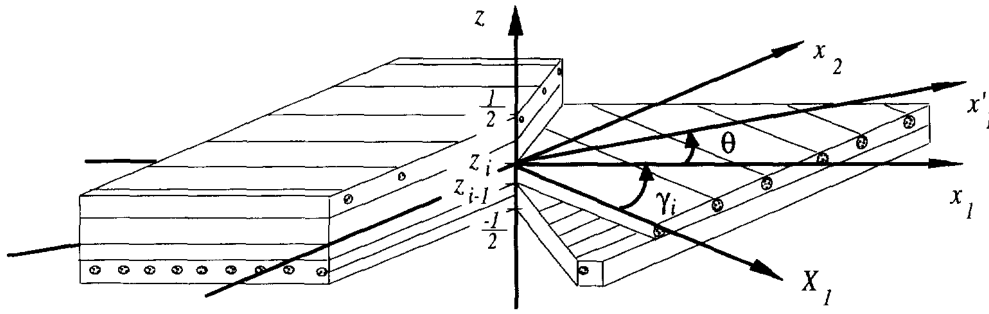

Fig. 1. Sketch of a laminate with the global coordinate systems x and x', a material system X and orientations of the plies shown.

$$\begin{Bmatrix} \{\mathbf{N}\} \\ \{\mathbf{M}\} \end{Bmatrix} = \begin{bmatrix} [\mathbf{A}] & [\mathbf{B}] \\ [\mathbf{B}] & [\mathbf{D}] \end{bmatrix} \begin{Bmatrix} \{\varepsilon^0\} \\ \{\kappa\} \end{Bmatrix} \tag{6}$$

where the $\sqrt{2}$-notation is used ($\{\mathbf{N}\} = \{\mathbf{N}_{11}\mathbf{N}_{22}\sqrt{2}\mathbf{N}_{12}\}^{\mathrm{T}}$ etc., see, e.g., Pedersen (1995)). The stiffness matrices for the whole laminate can be expressed in a very similar way as the constitutive matrix in eqn (2). The symmetric membrane, coupling and bending stiffness matrices [A], [B] and [D], respectively, are in terms of the material parameters C_{1-7} and the lamination parameters $\xi_{1-4}^{A,B,D}$ given as

$$\begin{aligned}
[\mathbf{A}] &= h([\boldsymbol{\Upsilon}_0]+[\boldsymbol{\Upsilon}_1]\xi_1^A+[\boldsymbol{\Upsilon}_2]\xi_2^A+[\boldsymbol{\Upsilon}_3]\xi_3^A+[\boldsymbol{\Upsilon}_4]\xi_4^A) \\
[\mathbf{B}] &= h^2([\boldsymbol{\Upsilon}_1]\xi_1^B+[\boldsymbol{\Upsilon}_2]\xi_2^B+[\boldsymbol{\Upsilon}_3]\xi_3^B+[\boldsymbol{\Upsilon}_4]\xi_4^B) \\
[\mathbf{D}] &= h^3(\tfrac{1}{12}[\boldsymbol{\Upsilon}_0]+[\boldsymbol{\Upsilon}_1]\xi_1^D+[\boldsymbol{\Upsilon}_2]\xi_2^D+[\boldsymbol{\Upsilon}_3]\xi_3^D+[\boldsymbol{\Upsilon}_4]\xi_4^D)
\end{aligned} \tag{7}$$

where the lamination parameters in a global coordinate-system x are defined as the weighted trigonometric integrals over the thickness:

$$\xi_{[1,2,3,4]}^{A,B,D} = \int_{-1/2}^{1/2} z^{0,1,2}\,[\cos 2\gamma(z), \cos 4\gamma(z), \sin 2\gamma(z), \sin 4\gamma(z)]\,\mathrm{d}z. \tag{8}$$

This compact notation implies, for instance, that ξ_3^B is given as

$$\xi_3^B = \int_{-1/2}^{1/2} z \sin 2\gamma(z)\,\mathrm{d}z. \tag{9}$$

In the following we will use the lamination parameters as the design variables in the design of the lay-up of a physical laminate. This implies that we are only dealing with at most twelve design variables per design domain in the plate (from any point in the plate to just the same lay-up throughout the plate). Moreover, the stiffnesses of the plate are linear in the design parameters. However, the parameters are not independent, enforcing an identification of the set of realizable lamination parameters. Also, the inverse problem of finding a lay-up which realizes an optimal combination of lamination parameters should be addressed. Both of these questions will be addressed in subsequent sections, and we shall especially be concerned with the simplifications which occur in the context of maximal stiffness design.

It should be emphasized that the lamination parameters of course depend on the chosen coordinate system. Therefore at least one of the twelve parameters can always be chosen as zero in an appropriate chosen reference system. Using simple trigonometric rules it is seen that the parameters in a coordinate system rotated the angle θ (see Fig. 1) are given by

$$\begin{Bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{Bmatrix}_{x'}^{A,B,D} = \begin{bmatrix} \cos 2\theta & 0 & -\sin 2\theta & 0 \\ 0 & \cos 4\theta & 0 & -\sin 4\theta \\ \sin 2\theta & 0 & \cos 2\theta & 0 \\ 0 & \sin 4\theta & 0 & \cos 4\theta \end{bmatrix} \begin{Bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{Bmatrix}_{x}^{A,B,D}. \tag{10}$$

We close this section by noting that the plate in the membrane state can be regarded as just one single layer of anisotropic material with the laminate material properties C_{1-7}^{lam} based on the laminate constitutive matrix $[\mathbf{C}]^{lam}$ (which is in most cases not equal to the former ply material $[\mathbf{C}]$). Hereby, the following relations exist between the first four lamination parameters ξ_{1-4}^A, the *material* parameters C_{1-7} and the *laminate* material parameters C_{1-7}^{lam}:

$$\begin{aligned} C_1^{lam} &= C_1 \\ C_2^{lam} &= C_2\xi_1^A + 2C_6\xi_3^A \\ C_3^{lam} &= C_3\xi_2^A + C_7\xi_4^A \\ C_4^{lam} &= C_4 \\ C_5^{lam} &= C_5 \\ C_6^{lam} &= -\tfrac{1}{2}C_2\xi_3^A + C_6\xi_1^A \\ C_7^{lam} &= -C_3\xi_4^A + C_7\xi_2^A. \end{aligned} \tag{11}$$

Therefore, parallel to the eqns (4)–(5) the following equality is true if the *laminate itself* is actually orthotropic in some coordinate-system x' with respect to the in-plane stiffness

$$(C_7\xi_2^A - C_3\xi_4^A)((C_2\xi_1^A + 2C_6\xi_3^A)^2 - 4(C_6\xi_1^A - \tfrac{1}{2}C_2\xi_3^A)^2) \\ -4(C_6\xi_1^A - \tfrac{1}{2}C_2\xi_3^A)(C_3\xi_2^A + C_7\xi_4^A)(C_2\xi_1^A + 2C_6\xi_3^A) = 0. \tag{12}$$

Also, the direction of orthotropy is given by

$$\tan 2\theta = \frac{C_2\xi_3^A - 2C_6\xi_1^A}{C_2\xi_1^A + 2C_6\xi_3^A}, \quad \tan 4\theta = \frac{C_3\xi_4^A - C_7\xi_2^A}{C_3\xi_2^A + C_7\xi_4^A}. \tag{13}$$

That is, a laminate made out of many (perhaps anisotropic) plies can possess exactly the same membrane stiffness characteristics as if it is made out of a single rotated equivalent orthotropic layer of the same total thickness. In the case of an orthotropic ply material eqns (12) and (13) simplify to

$$C_2^2C_3(-(\xi_1^A)^2\xi_4^A + (\xi_3^A)^2\xi_4^A + 2\xi_1^A\xi_2^A\xi_3^A) = 0 \tag{14}$$

and

$$\tan 2\theta = \frac{\xi_3^A}{\xi_1^A}, \quad \tan 4\theta = \frac{\xi_4^A}{\xi_2^A}. \tag{15}$$

By analogy to the former, the corresponding expressions for the coupling stiffness matrix [B] and the bending stiffness matrix [D] to be orthotropic are obtained by exchanging the ξ^A_{1-4}'s to ξ^B_{1-4} or ξ^D_{1-4}, respectively.

3. GENERALIZED LAMINATION PARAMETERS

In the following we will consider lamination parameters arising from any arbitrary variations of the ply angles through the thickness of the plate, including limits of rapidly varying oscillations. Thus at any cross sectional position z, the plate can consist of infinitely many, infinitely thin plies with varying angles (chattering designs in the out of plane direction). For the laminates we allow for a micro-structural lay-up at each macroscopic position z. We thus extend the definition of the lamination parameters to

$$\xi^{A,B,D}_{[1,2,3,4]} = \int_{-1/2}^{1/2} z^{0,1,2}\mathbf{P}_{[1,2,3,4]}(z)\,\mathrm{d}z. \tag{16}$$

Here, $\mathbf{P}$ is a vector of weighted trigonometric functions

$$\mathbf{P}_{[1,2,3,4]}(z) = \int_0^\pi \rho_z(\gamma)[\cos 2\gamma, \cos 4\gamma, \sin 2\gamma, \sin 4\gamma]\,\mathrm{d}\gamma$$

$$\mathbf{P}_0(z) = \int_0^\pi \rho_z(\gamma)\,\mathrm{d}\gamma = 1 \tag{17}$$

corresponding to a microscopic lay-up defined by the density $\rho_z(\gamma)$ of the plies oriented the angle γ at the thickness position z (i.e., the laminate is locally at the position z given as a lay-up of layers of density $\rho_z(\gamma)$ at angle γ, with total density of layers being one), see Appendix.

In seminal work, Grenestedt and Gudmundson (1993), employed the use of chattering sequences to demonstrate that the set of lamination parameters **D** constitutes a convex and compact set in $\mathbb{R}^{12}$. These properties are automatically expressed in the representation given by eqns (16) and (17).

The advantage of expressing laminate plate design in terms of the lamination parameters is that one obtains a reduction in the number of variables to twelve (per point or per design area), irrespective of the number of plies. Moreover, one avoids a troublesome optimization over periodic functions of the rotation angles, as well as working with a discrete number of plies. The convexity of the set of lamination parameters together with the linear dependence of the stiffness on these parameters, implies further considerable simplification with respect to the basic mathematical structure of the problem.

The lamination parameters are not independent, as there exist trigonometric relations between the functions over which the weights are taken. We first note that the range of admissible weights $\mathbf{P}(z)$ are given by the solution to the geometric moment problem as given in Krein and Nudelman (1977). We thus have that $\mathbf{P}(z)$ is an $\mathscr{L}^\infty$-map from the interval $[-\frac{1}{2},\frac{1}{2}]$ to the set **H** defined by

$$\mathbf{H} = \left\{\mathbf{y}\in\mathbb{R}^4 \middle| \begin{array}{c} 2y_1^2(1-y_2)+2y_3^2(1+y_2)+y_2^2+y_4^2-4y_1y_3y_4 \leqslant 1 \\ y_1^2+y_3^2 \leqslant 1, \quad -1\leqslant y_2 \leqslant 1 \end{array}\right\}. \tag{18}$$

The set **H** is the convex hull (bounded by the supporting hyperplanes) of the closed curve $(\cos 2\gamma, \cos 4\gamma, \sin 2\gamma, \sin 4\gamma)$, $\gamma\in[0,\pi]$ in $\mathbb{R}^4$. We can thus conclude that the set **D** is also compact as well as convex in $\mathbb{R}^{12}$.

The constraints on $\mathbf{P}(z)$ are inherited directly by the lamination parameters $\xi^A_{[1,2,3,4]}$ governing the membrane stiffness, so we have that

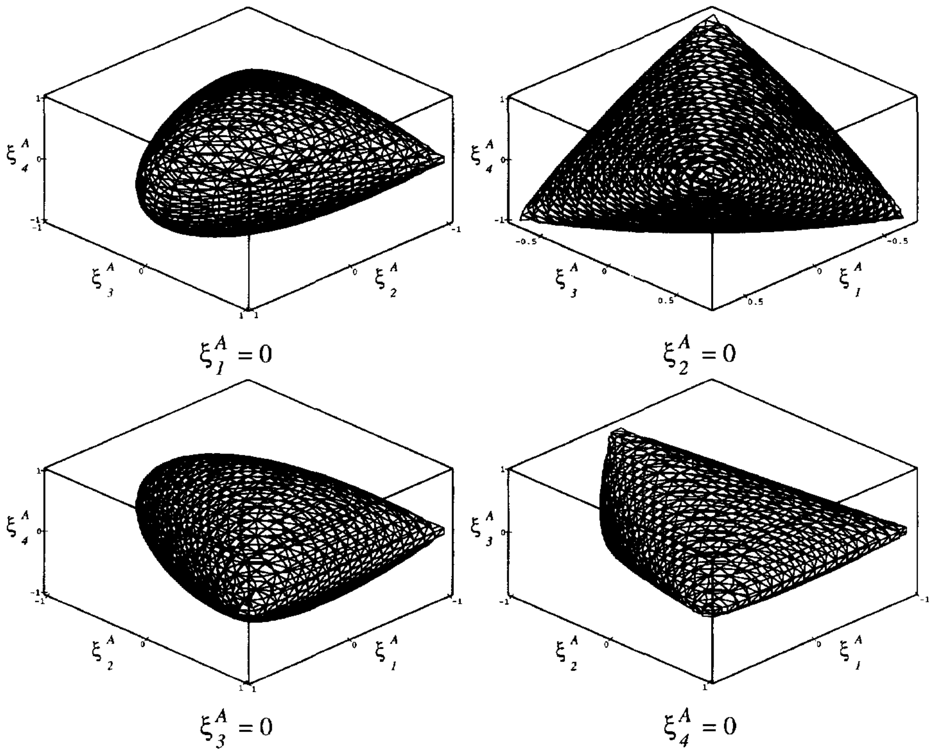

Fig. 2. The design domains for the four lamination parameters with one parameter equal to zero in turn.

$$
\begin{aligned}
2(\xi_1^A)^2(1-\xi_2^A)+2(\xi_3^A)^2(1+\xi_2^A)+(\xi_2^A)^2+(\xi_4^A)^2-4\xi_1^A\xi_3^A\xi_4^A &\leqslant 1 \\
(\xi_1^A)^2+(\xi_3^A)^2 &\leqslant 1 \\
-1 \leqslant \xi_2^A &\leqslant 1. \qquad (19)
\end{aligned}
$$

Figure 2 shows the feasible region for $\xi^A_{[1,2,3,4]}$, shown in $\mathbb{R}^3$ with one of each of the parameters set equal to zero in turn (cf. remark above about rotating to obtain zero parameters).

The same conditions as (19) holds for the four bending parameters $\xi^D_{1\text{–}4}$ when pure bending is considered. Many authors, Grenestedt and Gudmundson (1993) and Fukunaga and Vanderplaats (1991), have suggested some necessary conditions for different combinations of ξ's, but the complete set of sufficient conditions for all twelve parameters is still not known. As we shall see in the following, the full characterization is actually not required for design for maximum stiffness (minimum compliance). Moreover, for this case, the optimal lay-ups can be achieved without chattering designs.

In general the solution to the problem of finding a combination of ply thicknesses t_i's and ply angles γ_i's for prescribed lamination parameters ξ's is not unique. The problem can therefore be formulated and solved as an inverse optimization problem with the possibility of adding additional constraints as for example on the total thickness and/or on the variation of the orientation from ply to ply.

Regarding the number of plies, it can be shown that when only the $\xi^A_{1\text{–}4}$ governing the membrane stiffness are considered, a three ply laminate is needed but also sufficient to realize all points within the feasible domain given by eqn (19). Lipton (1994b) has developed an analytical method for finding the configuration (t_i's and γ_i's) of three ply laminates for any given $\xi^A_{1\text{–}4}$. In short, this is done by first rotating the set of parameters to a coordinate system in which one of these is zero, the angle of rotation found by eqn (10). Thereby the feasible domain in $\mathbb{R}^3$ looks like that shown in Fig. 2. A point within the domain is then expressed as a linear combination of a corner point and a point on the boundary. The corner point can be given by a single rotated ply whereas all points on the boundary can be obtained using only two plies. The needed three ply lay-up is obtained by combining the two former configurations, then finally rotating it back to the original coordinate system. See Lipton (1994b) for the complete derivation. We remark that an elegant result of Avellaneda and Milton (1989) can be used to show that the boundary points correspond to two plies. However, their approach is non-constructive and does not provide a means for identifying laminate configurations on the boundary.

4. MINIMUM COMPLIANCE

In this section the problem of minimizing the compliance of a laminate plate will be formulated in a form suitable for application of the lamination parameters as design variables. In the case of L multiple independent load cases the objective is to minimize a weighted sum of the compliances of each load case. The design variables are the lamination parameters varying from point to point throughout the plate. The lamination parameters are restricted to the feasible region **D** and in the case of a pure membrane problem **D** is given by eqn (19). The minimization problem is thus formulated as

$$\min_{\xi\in\mathbf{D}} \sum_{l=1}^{L} w_l W_l(v_l^*) \tag{20}$$

where w_l is the weight factor and W_l is the compliance given by the displacement field at equilibrium v_l^* of the load case l. Using Clayperons theorem with the assumption of dead loads (loads independent of displacements) and linear elasticity together with the principle of minimum potential energy, eqn (20) becomes

$$\min_{\xi\in\mathbf{D}} \sum_{l=1}^{L} w_l W_l(v_l^*) = -2\max_{\xi\in\mathbf{D}} \min_{v_l\in\mathbf{S}} \sum_{l=1}^{L} w_l(U_l(\{\varepsilon^0\}_l, \{\kappa\}_l) - W_l(v_l)). \tag{21}$$

The mid-plane strains and curvatures depend on the in plane and out of plane displacements, respectively. **S** is the space of kinematically admissible displacement fields and U_l is the total strain energy for the load case l given by

$$U_l(\{\varepsilon^0\}_l, \{\kappa\}_l) = \int_\Omega \frac{1}{2h}(\{\varepsilon^0\}_l^T[\mathbf{A}]\{\varepsilon^0\}_l + 2\{\varepsilon^0\}_l^T[\mathbf{B}]\{\kappa\}_l + \{\kappa\}_l^T[\mathbf{D}]\{\kappa\}_l)\,d\Omega \tag{22}$$

where $\int_\Omega d\Omega$ denotes integration over the midplane region Ω.

We remark that the strain energy is linear (and thus concave) in the lamination parameters. Furthermore, the lamination parameters are restricted to a bounded, closed and convex set. This, together with the convexity of the weighted sum of the potential energies as a function of the displacements implies that eqn (21) satisfies the conditions for existence of a saddle point (see, e.g., Lipton (1994a)) and the maximization and minimization operations can be interchanged. With the dead loads being also design independent the problem can be reformulated as

$$\max_{\xi\in\mathbf{D}} \min_{v_l\in\mathbf{S}} \sum_{l=1}^{L} w_l(U_l(\{\varepsilon^0\}_l, \{\kappa\}_l) - W_l(v_l)) = \min_{v_l\in\mathbf{S}}\left(\max_{\xi\in\mathbf{D}} \sum_{l=1}^{L} w_l U_l(\{\varepsilon^0\}_l, \{\kappa\}_l) - \sum_{l=1}^{L} w_l W_l(v_l)\right). \tag{23}$$

A similar operation was used in Bendsøe *et al.* (1995) for the analogous study of optimization of compliance with a free parametrization of material properties.

The saddle point argument employed above shows that we have existence of solutions to the minimum compliance problem without in plane chattering problems (see also the Appendix for an alternative proof of existence). Moreover, we note that the displacement field of the optimal plate is unique. This follows from the strict convexity of each energy appearing in the inner maximization of (23) (such a property was also noted by Petersson (1995), in his treatment of design of variable thickness sheets with contact). However, as we shall see later, the optimal designs may be non-unique and this complicates the analysis.

5. MAXIMIZATION OF THE STRAIN ENERGY USING LAMINATION PARAMETERS

We will here study the inner problem in the formulation (23), i.e., the maximization of the weighted potential energies with respect to the lamination parameters and for a fixed displacement field. As the loads are assumed to be design independent, this implies that we should solve the problem of maximizing the weighted strain energies

$$\max_{\substack{\xi\in\mathbf{D}\\ x\in\mathbf{\Omega}}} \sum_{l=1}^{L} w_l U_l(\{\boldsymbol{\varepsilon}^0\}_l, \{\boldsymbol{\kappa}\}_l). \tag{24}$$

We assume here that the lamination parameters can be chosen independently from point to point. As the strain energy densities are all positive, the maximization (24) therefore requires that the optimal lamination parameters maximize the pointwise weighted strain energy densities throughout the plane, i.e., the optimal lamination parameters $\xi\,(x_1, x_2)$ are in any point $\mathbf{x}\in\mathbf{\Omega}$ given as the solution to

$$\max_{\xi\in\mathbf{D}} \sum_{l=1}^{L} w_l u_l \tag{25}$$

with

$$u_l = \frac{1}{2h}(\{\boldsymbol{\varepsilon}^0\}_l^{\mathrm{T}}[\mathbf{A}]\{\boldsymbol{\varepsilon}^0\}_l + 2\{\boldsymbol{\varepsilon}^0\}_l^{\mathrm{T}}[\mathbf{B}]\{\boldsymbol{\kappa}\}_l + \{\boldsymbol{\kappa}\}_l^{\mathrm{T}}[\mathbf{D}]\{\boldsymbol{\kappa}\}_l). \tag{26}$$

We note here that the objective function of problem (24) is linear and that the constraint set is convex and compact. There thus exists a solution among the extreme points of the convex set $\mathbf{D}$. Here we can actually elaborate further to obtain certain properties of the solution. Also, we will see that the full characterization of $\mathbf{D}$ is not required.

We thus proceed to give the precise algebraic relation between the lamination parameters and the strain energy. The energy density can also be written directly in terms of the total strains $\{\boldsymbol{\varepsilon}(z)\}_l = \{\boldsymbol{\varepsilon}^0\}_l + zh\{\boldsymbol{\kappa}\}_l$ and using the matrix definitions in eqn (2) as

$$u_l = \frac{1}{2}\int_{-1/2}^{1/2}\left(\sum_{i=0}^{4} \{\boldsymbol{\varepsilon}(z)\}_l^{\mathrm{T}}[\boldsymbol{\Upsilon}_i]\{\boldsymbol{\varepsilon}(z)\}_l \mathbf{P}_i(z)\right)\mathrm{d}z. \tag{27}$$

As we also allow for any variation of the ply lay-up through the thickness of the plate, we see that in order to solve (25) we have for each position z through the thickness to maximize the expression

$$\sum_{l=1}^{L} w_l\left(\sum_{i=0}^{4} \{\boldsymbol{\varepsilon}(z)\}_l^{\mathrm{T}}[\boldsymbol{\Upsilon}_i]\{\boldsymbol{\varepsilon}(z)\}_l \mathbf{P}_i(z)\right) \tag{28}$$

over the parameters $\mathbf{P}_i$. As $\mathbf{P}_0 = 1$, we conclude that we can find the optimal lay-up of the laminate (for maximum stiffness) for each position (x_1, x_2, z) in the plate domain by solving the problem

$$\max_{\mathbf{y}\in\mathbf{H}} \sum_{l=1}^{L} w_l\left(\sum_{i=1}^{4} \{\boldsymbol{\varepsilon}(z)\}_l^T[\boldsymbol{\Upsilon}_i]\{\boldsymbol{\varepsilon}(z)\}_l y_i\right) \tag{29}$$

over the well-known set **H** given earlier. Problem (29) is also, like problem (24), a linear optimization problem with a convex and compact constraint set. There thus exists a solution among the extreme points of the convex set **H**. For problem (29) we know that the constraint set **H** is the convex hull of the curve $(\cos 2\gamma, \cos 4\gamma, \sin 2\gamma, \sin 4\gamma)$, $\gamma \in [0, \pi]$, so we conclude that for each position (x_1, x_2, z) in the plate there exists a solution to the local problem (29) which corresponds to a single ply rotated at a given angle. However, as this solution is not unique, we cannot make this conclusion about the optimal design. The optimal design will be governed by the requirement that the unique optimal displacement together with an appropriate solution to (29) should satisfy equilibrium under the given load(s). This point will be elaborated later for the pure membrane case. However, one further consideration can be drawn here regarding the nature of the optimal solution. Consider the design parameters parametrized by an angle of rotation of the total laminate, together with the lamination parameters $(y_1, y_2, y_3, 0)$ in **H** (cf. the discussion in Section 2). The linearity of the rigidity in the three parameters (y_1, y_2, y_3) ensures that any optimal design must lie on the surface of the admissible set equal to the $\xi_4^A = 0$-domain illustrated in Fig. 2. It can be shown that this surface consists of two-ply laminates (cf. the discussion in Section 3), so we conclude that an optimal design will always consist of at most two plies per thickness position, in a chattering design.

Note that problem (29) is very tractable from a computational point of view, being a convex optimization problem in only four variables. It is thus viable to solve this problem numerically for any element of even a fine discretization of the plate. We will return to these issues in a later section.

We remark here that the analysis above carries over almost ad verbatim for a number of other design scenarios. First, consider the case where only a finite set of domains of the plate can be designed independently. The ultimate variation of this is the case where all of the plate is supposed to consist of the same lay-up. Then the optimal lay-up in the design area Ω_k at the position z along the normal to the plate is given as the solution to the convex optimization problem

$$\max_{\mathbf{y}\in\mathbf{H}} \sum_{l=1}^{L} w_l\left(\sum_{i=1}^{4}\left(\int_{\Omega_k} \{\boldsymbol{\varepsilon}(z)\}_l^T[\boldsymbol{\Upsilon}_i]\{\boldsymbol{\varepsilon}(z)\}_l \,\mathrm{d}\Omega\right) y_i\right) \tag{30}$$

where $\int_{\Omega_k} \{\boldsymbol{\varepsilon}(z)\}_l^T[\boldsymbol{\Upsilon}_i]\{\boldsymbol{\varepsilon}(z)\}_l \,\mathrm{d}\Omega$ is an average energy-like expression, taken over the design area Ω_k. A further restriction in the design freedom can be imposed by requiring that the plate through its thickness is required to have the same lay-up in a number of intervals defined in normalized coordinates as $[z_m, z_{m+1}]$, $-\frac{1}{2} = z_0 < z_1 < \ldots < z_m = \frac{1}{2}$. Then the optimal lay-up in the design area Ω_k and the thickness interval $[z_m, z_{m+1}]$ will be given as the solution of the convex optimization problem

$$\max_{\mathbf{y}\in\mathbf{H}} \sum_{l=1}^{L} w_l\left(\sum_{i=1}^{4}\left(\int_{z_m}^{z_{m+1}}\int_{\Omega_k} \{\boldsymbol{\varepsilon}(z)\}_l^T[\boldsymbol{\Upsilon}_i]\{\boldsymbol{\varepsilon}(z)\}_l \,\mathrm{d}\Omega\,\mathrm{d}z\right) y_i\right) \tag{31}$$

where we now have at most two angles of rotation of an out of plane chattering two ply laminate for each design area and thickness interval.

Further simplifications are possible if we consider single loads or cases where only membrane or bending effects appear; especially useful for the membrane case where chattering designs can be avoided.

6. THE PURE MEMBRANE CASE

Let us in this section consider the situation of designing the lay-up for a situation of only in plane loading, i.e., the pure membrane case. In that setting the strain energy density of the plate reduces to

$$u_l = \frac{1}{2}\int_{-1/2}^{1/2}\left(\sum_{i=0}^{4}\{\boldsymbol{\varepsilon}^0\}_l^{\mathrm{T}}[\boldsymbol{\Upsilon}_i]\{\boldsymbol{\varepsilon}^0\}_l\mathbf{P}_i(z)\right)\mathrm{d}z. \tag{32}$$

Thus the optimization over the variables $\mathbf{P}_i$ gives the same result at any cross-sectional position z of the plate. Together with the fact that the stacking sequence is of no consequence for the membrane stiffness and the arguments above for the general situation, this implies that the optimal plate can be constructed from at most two plies for each design area of the plate. The design may be any angle ply with different ply thicknesses rotated at a given angle. This holds for the single as well as the multiple load case. In the single load case further information can be obtained, as will be shown below.

The optimal angle for the single ply realization for a variation of design in a design domain Ω_k is given as a maximizer of the functional

$$\begin{aligned}\Phi(\gamma) &= \Psi_1\cos 2\gamma+\Psi_2\cos 4\gamma+\Psi_3\sin 2\gamma+\Psi_4\sin 4\gamma\\ \Psi_i &= \sum_{l=1}^{L} w_l\int_{\Omega_k}\{\boldsymbol{\varepsilon}^0\}_l^{\mathrm{T}}[\boldsymbol{\Upsilon}_i]\{\boldsymbol{\varepsilon}^0\}_l\,\mathrm{d}\Omega,\quad i=1,\ldots,4.\end{aligned} \tag{33}$$

For orthotropic materials and with a pointwise design variation, this problem has been solved analytically for the single load case in Seregin and Troiskii (1982), Fedorov and Cherkaev (1983) and in Pedersen (1989, 1990a), while the multiple load case is treated by analogous means in Díaz and Bendsøe (1992). From the expression (33) it can be readily seen that the developments in the latter reference carries over directly to the present situation, so that the stationary points of the functional (33) can be identified by solving analytically a fourth order polynomial in for example $\sin 2\gamma$. As indicated here, the works mentioned employ the rotation angle for the analytical studies. However, problem (33) can also be seen as problem in the lamination parameters. Thus problem (33) is equivalent to considering the problem

$$\max_{\mathbf{y}\in\mathbf{H}}\left(\sum_{i=1}^{4}\Psi_i y_i\right) \tag{34}$$

in the sense that there will be solutions to (34) which correspond to a rotation angle (as noted earlier, the optimization of the linear functional in (34) over the convex hull $\mathbf{H}$ of the curve $(\cos 2\gamma, \cos 4\gamma, \sin 2\gamma, \sin 4\gamma)$, $\gamma\in[0,\pi]$ will result in at least one solution on this curve). If considering numerical optimization procedures, the convex problem (34) is to be preferred over maximizing the periodic functional (33).

Let us now use the lamination parameters to solve (34) for the case of a single load case, an orthotropic material and the pointwise variation of design. This solution (in terms of the optimal value of (34)) will then be combined with the outer minimization in (23) to obtain information on the character of the optimal solution to the minimum compliance problem, in contrast to only considering the local problem for fixed strains.

For simplicity, in our point of interest we use the directions of principal strains, ε_I, ε_{II}, as a local frame of reference, while for the ply material we assume that the directions of orthotropy are ordered so $C_2 \geqslant 0$, i.e., so $C_{1111} \geqslant C_{2222}$. The lamination parameters will then characterize the rotation of the orthotropic plies relative to the directions of principal strains of the single load. In a given point we thus have $C_6 = 0$, $C_7 = 0$ and $\{\boldsymbol{\varepsilon}^0\}_l^{\mathrm{T}} = \{\varepsilon_I, \varepsilon_{II}, 0\}$, and problem (34) then reduces to

$$\max_{\mathbf{y} \in \mathbf{H}} (C_2(\varepsilon_I^2 - \varepsilon_{II}^2) y_1 + C_3(\varepsilon_I - \varepsilon_{II})^2 y_2). \tag{35}$$

Furthermore, as the third and fourth lamination parameters do not enter in (35), we can reduce the constraint set to the range of the trigonometric averages $\mathbf{P}_1$, $\mathbf{P}_2$, that is, to the set $\hat{\mathbf{H}} = \{\mathbf{y} \in R^2 | -1 \leqslant y_1 \leqslant 1, -1 \leqslant y_2 \leqslant 1, 2y_1^2(1-y_2)+y_2^2 \leqslant 1,\}$, so that (35) is reduced to

$$\max_{[y_1, y_2] \in \hat{\mathbf{H}}} (C_2(\varepsilon_I^2 - \varepsilon_{II}^2) y_1 + C_3(\varepsilon_I - \varepsilon_{II})^2 y_2). \tag{36}$$

Assume first that $C_3 \geqslant 0$; this is a material which we say has low shear stiffness, cf. Pedersen (1990). As $C_3(\varepsilon_I - \varepsilon_{II})^2 \geqslant 0$, $C_2 \geqslant 0$, the optimal energies will depend on the sign of $(\varepsilon_I^2 - \varepsilon_{II}^2)$ and will be given by the energies obtained from the lamination parameters $[y_1, y_2] = [1, 1]$ (if $(\varepsilon_I^2 - \varepsilon_{II}^2) \geqslant 0$) and $[y_1, y_2] = [-1, 1]$ (if $(\varepsilon_I^2 - \varepsilon_{II}^2) \leqslant 0$). If $(\varepsilon_I^2 - \varepsilon_{II}^2) \neq 0$, the design is unique and corresponds to a single ply rotated so the numerically largest principal strain is aligned with the material axis corresponding to C_{1111} (we have assumed $C_{1111} \geqslant C_{2222}$). The optimal energy (i.e., the optimal value of (32)) becomes

$$\Phi(\{\boldsymbol{\varepsilon}^0\}) = \tfrac{1}{2}\max\{C_2(\varepsilon_I^2 - \varepsilon_{II}^2) + C_3(\varepsilon_I - \varepsilon_{II})^2, -C_2(\varepsilon_I^2 - \varepsilon_{II}^2) + C_3(\varepsilon_I - \varepsilon_{II})^2\} \tag{37}$$

which is non-smooth at strains which satisfy $\varepsilon_I^2 = \varepsilon_{II}^2$, i.e., uniform dilation or pure shear (in terms of strains). The resulting reduced minimum potential energy problem (the outer minimization problem in displacements of (23)) is then a non-smooth, convex problem, for which the necessary conditions of optimality at points with $\varepsilon_I^2 = \varepsilon_{II}^2$ will involve a convex combination of the gradients of the two smooth branches of Φ. This implies that at points where the strains of the optimal plate satisfy $\varepsilon_I^2 = \varepsilon_{II}^2$, the optimal design can consist of some cross-ply consisting of two plies rotated of 0 and 90 degrees relative to the principal strain axes, with thicknesses decided through the conditions of equilibrium.

Then consider the case $C_3 \leqslant 0$ (a material with high shear stiffness). Here the algebra becomes somewhat messier. In this case we get unique solutions to (34) if $\varepsilon_I \neq \varepsilon_{II}$, with the solution corresponding to a single ply rotated at an angle γ given by $\cos 2\gamma = -C_2(\varepsilon_I + \varepsilon_{II})/(4C_3(\varepsilon_I - \varepsilon_{II}))$ if $C_2(\varepsilon_I + \varepsilon_{II})/(4C_3(\varepsilon_I - \varepsilon_{II})) \leqslant 1$ and given as $\gamma = 0$ if $C_2(\varepsilon_I + \varepsilon_{II})/(4C_3(\varepsilon_I - \varepsilon_{II})) \geqslant 1$.

In the case $\varepsilon_I = \varepsilon_{II}$ we have a non-unique solution and the optimal energy becomes non-smooth, resulting in an optimal design which also in this case must consist of some cross-ply at 0 and 90 degrees relative to the principal axes. In every case the resulting optimal design is realized by an orthotropic laminate which is not necessarily the result in the multiple load case.

7. COMPUTATIONAL EXAMPLES

In this section are shown some results of the use of the lamination parameters in the optimization process.

A laminate built up of plies with the material data of graphite/epoxy taken from Tsai and Hahn (1980) is designed to maximize the stiffness. Only in plane loads are considered thereby having the stiffnesses given by the first four lamination parameters $\xi_1^A{}_4$. The total plate thickness is kept constant equal to one resulting in a purely local optimization problem. The deformations and strains of the plate are determined using the finite element method in which a set of lamination parameters is related to each finite element. Based on the strains and the problem formulation (34) the optimal design variables are then found in each finite element using a standard sequential quadratic programming (SQP) algorithm, Schittkowski (1986). These are then used in a new finite element analysis and so on until convergence is reached. To avoid the problem discussed previously of non-unique solutions to the inner max-problem of optimizing the local strain energy, the objective function has

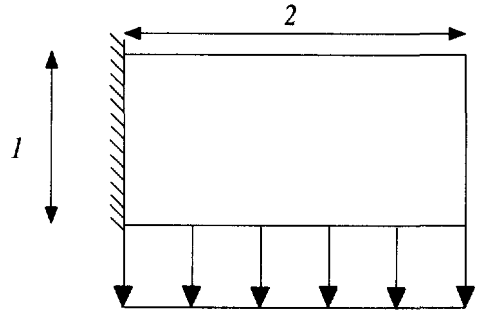

Fig. 3. A cantilever subjected to a vertical load.

been made strictly concave in the lamination parameters by adding an explicit penalty term in these parameters, yielding a modified inner problem:

$$\max_{\xi_i^A \in \mathbf{H}} \left(\sum_{i=1}^{4} \Psi_i \xi_i^A - \sum_{i=1}^{4} (\xi_i^A)^2 \epsilon \right). \tag{38}$$

For a problem satisfying the conditions for existence of a saddlepoint, as in (23), it can be shown (cf. Ekeland and Temam (1976), Fortin and Glowinski (1983)) that the solution to the penalized problem converges to the optimal solution of the original un-penalized problem for $\epsilon \to 0$ (the solution to our saddle formulation of the minimum compliance problem is thus numerically obtained by what is usually called a 'viscosity approach', see, e.g., Ekeland and Temam (1976), Fortin and Glowinski (1983). In the implemented algorithm the variable ϵ is initially set to one and then gradually decreased to zero through the iterations.

In the first example only a single load case is considered. The model is a cantilever fixed at one end and being loaded with a uniform distributed vertical load as sketched in Fig. 3. The method of Lipton (1994b) was applied after the iteration process to find a laminate lay-up with the optimal properties. This algorithm generates a three ply laminate, as this is the fewest number of plies needed to obtain all possible combinations of lamination parameters. It should be emphasized that this lay-up however is not in general unique. The resulting three ply design is shown in Fig. 4 and we see that the numerical solution reflects the theoretical results stated earlier. The hatch direction in each element marks the orientation of the fibers, the colour shows the ply thickness. The darker the colour the thinner the ply. As the loading is in-plane, the order of the plies is non-important. As can be seen, the solution is a rotated single ply in the outermost regions and a cross-ply with varying ply thicknesses in the center. To control that this is actually the case, the total energy was also calculated for this design based on the cross-ply angles and thicknesses and compared to the energy for the design before post-processing based on the lamination parameters yielding actually a slightly better result for the former. Furthermore equilibrium was tested by measuring the residuals from one equation to the next, i.e., the deformations from before a design change were multiplied to the stiffness matrix assembled after the design change and the forces (design independent) subtracted. In Fig. 5 is shown the strain situation in each finite element in the optimal design of Fig. 4. Black corresponds to pure dilation, $\varepsilon_I = \varepsilon_{II}$, and white to pure shear, $\varepsilon_I = -\varepsilon_{II}$. It is evident that these special cases which one might consider as rare exceptions actually end up being dominant after optimization.

The second example is a 'bridge'-structure simply supported at both ends and loaded with three independent point loads all given the same weight factors. The model is shown in Fig. 6.

The optimal three ply design is shown in Fig. 7. The same optimization procedure is applied as before. Again the solution is a two-ply solution as predicted by theory, but in contrast to the solution for a single load case the design is not made out of a cross-ply or a

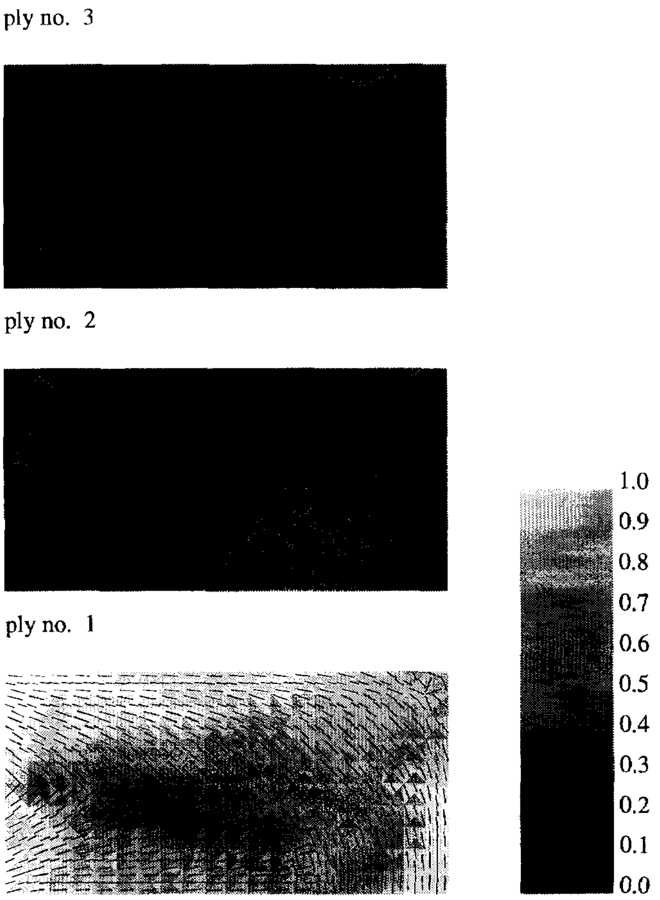

Fig. 4. Optimal three ply laminate.

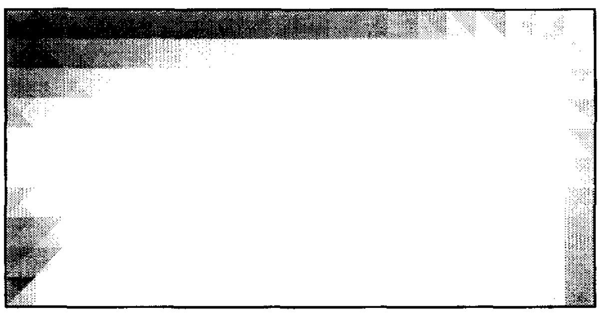

Fig. 5. $|\varepsilon_I - \varepsilon_{II}|/2|\varepsilon_I|$ for the optimal design of the cantilever.

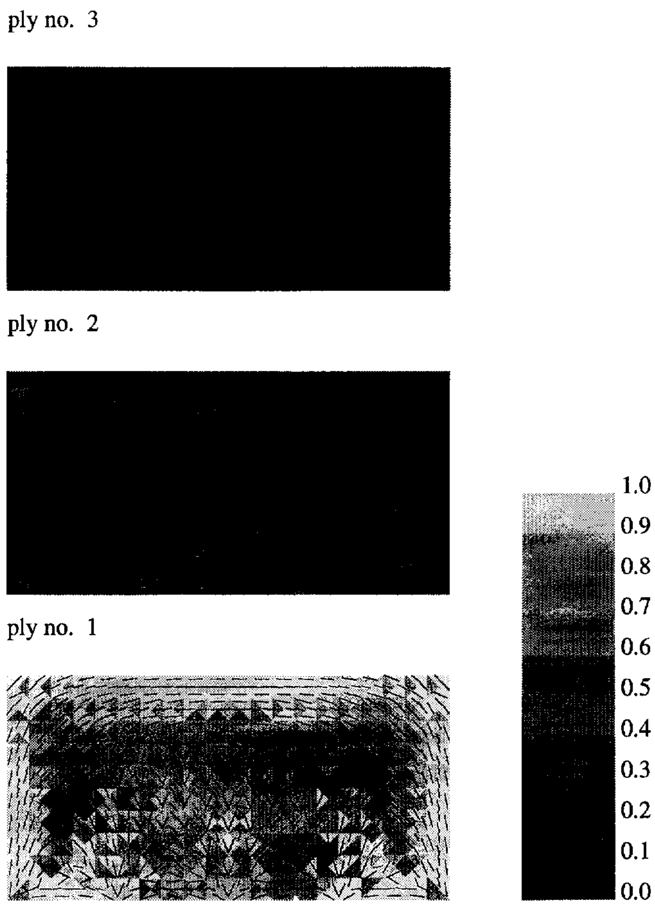

Fig. 7. Optimized three ply design.

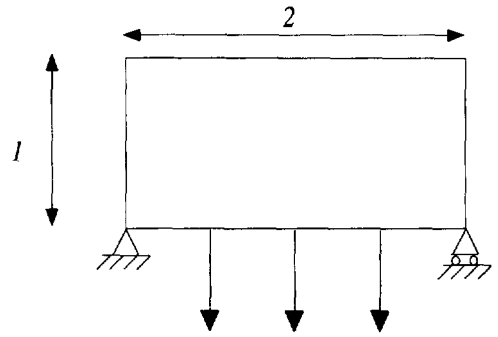

Fig. 6. A plate with three independent single loads applied.

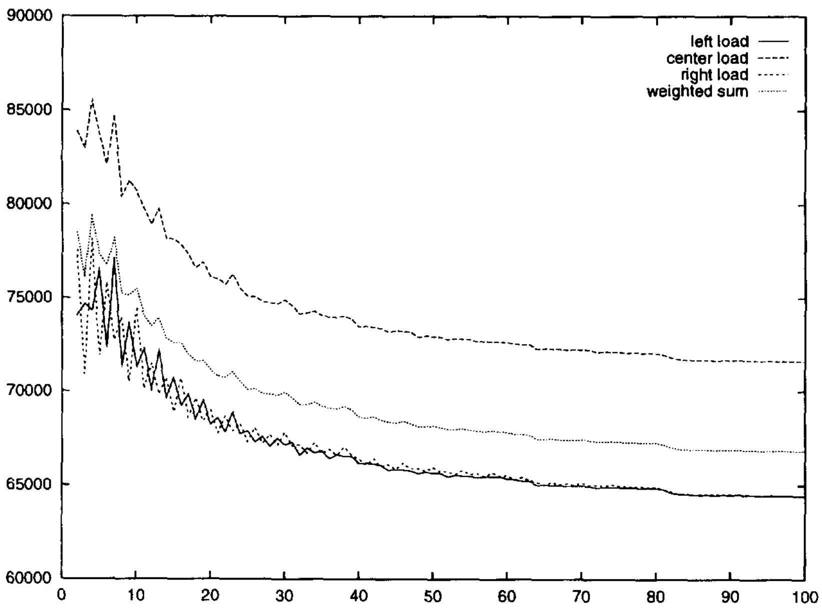

Fig. 8. The total energies for each of the three load cases and their weighted sum vs iteration number for the second example.

single ply in a number of elements. Neither is the laminate itself orthotropic (this which was controlled through the use of eqn (12)). The iteration history of the total strain energy vs iteration number is shown in Fig. 8. The relatively slow convergence is greatly extended due to the use of the penalty approach; this is then the price paid for avoiding algorithms for non-differentiable optimization.

Finally, one observes that when using lamination parameters, the optimization is independent of the starting design, i.e., one avoids the problem of local optima which is often troublesome when solving the problem directly in terms of fiber orientations.

8. CONCLUSION

The parametrization of lamination parameters has been thoroughly described in the context of classical laminate theory. The most general case of any number and combination of anisotropic plies subjected to multiple both in and out of plane loads has been examined. Relations between the material parameters and the lamination parameters as well as conditions for orthotropy of the laminate based on the lamination parameters have been derived.

In relation to stiffness optimization a vast simplification of the problem is obtained, as the number of design variables is shown to be reduced to one third of the original number. This result is based on the extension of the design space to include out of plane chattering designs, exploiting the characteristics of the feasible set for the design variables and the linear relation between stiffnesses and design parameters. The design problems for some special choices of design domains are also treated.

In the special case of pure membrane forces acting on the laminate an optimal solution can be realized as a laminate with at most two plies. Moreover, for a single load case, the optimal solution is a single layer of material or a cross-ply throughout the plate. Finally the parametrization leads to a computational advantage compared to solving the optimization problem directly in terms of the fiber orientations. This is illustrated with two examples.

Acknowledgements—The authors would like to thank J. L. Grenestedt and an anonymous reviewer for many helpful remarks and for pointing out an erroneous conclusion in the first version of this paper. This work was supported in part by the Danish Technical Research Council, through the Programme of Research on Computer Aided Design (VBH, MPB, PP). The support of the Danish Natural Sciences Research Council and the National Science Foundation through grant DMS-9205158 (RL) is also gratefully acknowledged.

REFERENCES

Allaire, G. and Kohn, R. V. (1993). Optimal design for minimum weight and compliance in plane stress using extremal microstructures. *Eur. J. Mech. A.* **12**(6), 839–878.

Avallaneda, M. and Milton, G. W. (1989). Bounds on the effective elasticity tensor composites based on two-point correlation. In *Composite Material Technology*, (eds Hui, D. and Kozik, T. J.), ASME, New York, pp. 89–93.

Ball, J. M. (1989). A version of the fundamental theorem for young measures. In *Partial Differential Equations and Continuum Models of Phase Transitions* (eds Rascal, M., Serre, D. and Slemrod, M.), Springer, Berlin, pp. 207–215.

Bendsøe, M. P. (1995). *Optimization of Structural Topology, Shape, and Material*, Springer, Berlin, Heidelberg.

Bendsøe, M. P., Díaz, A. R., Lipton, R. and Taylor, J. E. (1995). Optimal design of material properties and material distribution of multiple loading conditions. *Int. J. Num. Meth. Engng* **38**, 1149–1170.

Cea, J. and Malanovsky, K. (1970). An example of a max-min problem in partial differential equations. *SIAM J. Control* **8**, 305–316.

Cherkaev, A. and Palais, R. (1996). Optimal design of three-dimensional axisymmetric elastic structures. *Struct. Optim.*, to appear.

Díaz, A. R. and Bendsøe, M. P. (1992). Shape optimization of structures for multiple loading conditions using a homogenization method. *Struct. Optim.* **4**, 17–22.

Díaz, A. R., Lipton, R. and Soto, C. A. (1995). A new formulation of the problem of optimum reinforcement of Reissner-Mindlin plates. *Comp. Meth. Appl. Mech. Engng* **123**, 121–139.

Ekeland, I. and Temam, R. (1976). *Convex Analysis and Variational Problems*, North Holland, Amsterdam.

Fang, C. and Springer, G. S. (1993). Design of composite laminates by a Monte Carlo method. *J. Comp. Mat.* **27**, 721–753.

Fedorov, A. V. and Cherkaev, A. V. (1983). Choice of optimal orientation of axes of elastic symmetry for an orthotropic plate. *MTT* **18**, 135–142.

Fortin, M. and Glowinski, R. (1983). *Augmented Lagrangian Methods*, North-Holland, Amsterdam.

Fukunaga, H. and Vanderplaats, G. N. (1991). Stiffness optimization of orthotropic laminated composites using lamination parameters. *AIAA J.* **29**, 641–646.

Fukunaga, H. and Sekine, H. (1992). Stiffness design method of symmetric laminates using lamination parameters. *AIAA J.* **30**, 2791–2793.

Fukunaga, H. and Sekine, H. (1993). Optimum design of composite structures for shape, layer angle and layer thickness distributions. *J. Comp. Mat.* **27**, 1479–1492.

Fukunaga, H., Sekine, H. and Sato, M. (1994). Optimal design of symmetric laminated plates for fundamental frequency. *J. Sound Vibration* **171**, 219–229.

Fukunaga, H. and Sekine, H. (1994). A laminate design for elastic properties of symmetric laminates with extension-shear or bending-twisting coupling. *J. Comp. Mat.* **28**, 708–731.

Grenestedt, J. L. (1991). Layup optimization against buckling of shear panels. *Struct. Opt.* **3**, 115–120.

Grenestedt, J. L. and Gudmundson, P. (1993). Layup optimization of composite material structures. In *Optimal Design with Advanced Materials* (ed. Pedersen, P.), Elsevier Science, Amsterdam, 311–336.

Haftka, R. T., Gürdal, Z. and Kamat, M. P. (1990). *Elements of Structural Optimization*, Kluwer Academic Publishers, Dordrecht.

Jog, C., Haber, R. B. and Bendsøe, M. P. (1994). Topology design with optimized, self-adaptive materials. *Int. J. Num. Meth. Engng* **37**, 1323–1350.

Krein, M. G. and Nudelman (1977). The Markov moment problem and extremal problems. In *Translations of Mathematical Monographs*, Vol. 50, American Mathematical Society, Providence, Rhode Island.

Lipton, R. (1994a). A saddle point theorem with application to structural optimization. *JOTA* **81**, 549–568.

Lipton, R. (1994b). On optimal reinforcement of plates and choice of design parameters. *Control Cybernetics* **23**, 481–493.

Lurie, K. A. (1993). *Applied Optimal Control Theory of Distributed Systems*, Plenum Press, New York.

Miki, M. (1982). Material design of composite laminates with required in-plane elastic properties. In *Progress in Science and Engineering of Composites* (eds Hayashi, T., Kawata, K. and Umekawa, S.), ICCM-IV, Tokyo, pp. 1725–1731.

Miki, M. and Sugiyama, Y. (1993). Optimum design of laminated composite plates using lamination parameters. *AIAA J.* **31,** 921–922.

Pedersen, P. (1989). On optimal orientation of orthotropic materials. *Struct. Opt.* **1,** 101–106.

Pedersen, P. (1990a). Bounds on elastic energy in solids of orthotropic materials. *Struct. Opt.* **2,** 55–63.

Pedersen, P. (1990b). Combining material and element rotation in one formula. *Comm. Appl. Num. Meth.* **6,** 549–555.

Pedersen, P. (1995). Simple transformations by proper contracted forms. *Comm. Num. Meth. Engng* **11,** 821–829.

Pedersen, P. and Bendsøe, M. P. (1995). On strain-stress fields resulting from optimal orientation. In *WCSMO-1* (eds Olhoff, N. and Rozvany, G. I. N.), Pergamon, Oxford, 243–250.

Petersson, J. (1996) On stiffness maximization of variable thickness sheet with unilateral constraints. *Quart. Appl. Math.*, to appear.

Schittkowski, K. (1986). NLPQL: a Fortran subroutine solving constrained nonlinear programming problems. *Ann. Operations Res.* **5,** 485–500.

Seregin, G. A. and Troitskii, V. A. (1982). On the best position of elastic symmetry planes in an orthotropic body. *PMM USSR* **45,** 139–142.

Tsai, S. W. and Hahn, H. T. (1989). *Introduction to Composite Materials*, Technomic, Westport.

APPENDIX A. EXISTENCE OF SOLUTIONS OF STIFFNESS DESIGN PROBLEMS USING GENERALIZED LAMINATION PARAMETERS AND CHATTERING DESIGNS

We demonstrate that the weighted compliances given by eqn (20) are $\mathscr{L}^\infty$ weak $*$ lower semicontinuous in the generalized lamination parameters; this serves to establish existence for minimum stiffness design. Our proof is classical and follows the observations of Cea and Malanovsky (1970).

We consider a sequence of generalized lamination parameters $\{\xi^{A,B,D}_{[1,2,3,4]}j\}_{j=1}^{\infty}$ converging in $\mathscr{L}^\infty[0,\pi]^{12}$ weak $*$ to $\xi^{A,B,D}_{[1,2,3,4]_\infty}$. The associated sequence of displacement fields are written $\{v_l^{*j}\}_{j=1}^{\infty}$ and $v_l^{*\infty}$. One has

$$\begin{aligned}
&\sum_{l=1}^{L} w_l W_l(v_l^{*j}) - \sum_{l=1}^{L} w_l W_l(v_l^{*\infty}) \\
&= \sum_{l=1}^{L} w_l W_l(v_l^{*j}) - 2\sum_{l=1}^{L} w_l W_l(v_l^{*\infty}) + \sum_{l=1}^{L} w_l W_l(v_l^{*\infty}) \\
&= \sum_{l=1}^{L} w_l \int_\Omega \frac{1}{2h}(\{\boldsymbol{\varepsilon}_j^0 - \boldsymbol{\varepsilon}_\infty^0\}_l^{\mathrm{T}}[\mathbf{A}_j]\{\boldsymbol{\varepsilon}_j^0 - \boldsymbol{\varepsilon}_\infty^0\}_l + 2\{\boldsymbol{\varepsilon}_j^0 - \boldsymbol{\varepsilon}_\infty^0\}_l^{\mathrm{T}}[\mathbf{B}_j]\{\boldsymbol{\kappa}_j^0 - \boldsymbol{\kappa}_\infty^0\}_l \\
&\quad + \{\boldsymbol{\kappa}_j^0 - \boldsymbol{\kappa}_\infty^0\}_l^{\mathrm{T}}[\mathbf{D}_j]\{\boldsymbol{\kappa}_j^0 - \boldsymbol{\kappa}_\infty^0\}_l)\,\mathrm{d}\Omega \\
&\quad + \sum_{l=1}^{L} w_l \int_\Omega \frac{1}{2h}(\{\boldsymbol{\varepsilon}_\infty^0\}_l^{\mathrm{T}}[\tilde{\mathbf{A}}]\{\boldsymbol{\varepsilon}_\infty^0\}_l + 2\{\boldsymbol{\varepsilon}_\infty^0\}_l^{\mathrm{T}}[\tilde{\mathbf{B}}]\{\boldsymbol{\kappa}_\infty^0\}_l + \{\boldsymbol{\kappa}_\infty^0\}_l^{\mathrm{T}}[\tilde{\mathbf{D}}]\{\boldsymbol{\kappa}_\infty^0\}_l)\,\mathrm{d}\Omega
\end{aligned} \tag{A1}$$

where

$$\begin{aligned}
[\tilde{\mathbf{A}}] &= h([\boldsymbol{\Upsilon}_0] + [\boldsymbol{\Upsilon}_1](\xi^A_{1\infty} - \xi^A_{1j}) + [\boldsymbol{\Upsilon}_2](\xi^A_{2\infty} - \xi^A_{2j}) + [\boldsymbol{\Upsilon}_3](\xi^A_{3\infty} - \xi^A_{3j}) + [\boldsymbol{\Upsilon}_4](\xi^A_{4\infty} - \xi^A_{4j})) \\
[\tilde{\mathbf{B}}] &= h^2([\boldsymbol{\Upsilon}_1](\xi^B_{1\infty} - \xi^B_{1j}) + [\boldsymbol{\Upsilon}_2](\xi^B_{2\infty} - \xi^B_{2j}) + [\boldsymbol{\Upsilon}_3](\xi^B_{3\infty} - \xi^B_{3j}) + [\boldsymbol{\Upsilon}_4](\xi^B_{4\infty} - \xi^B_{4j})) \\
[\tilde{\mathbf{D}}] &= h^3(\tfrac{1}{12}[\boldsymbol{\Upsilon}_0] + [\boldsymbol{\Upsilon}_1](\xi^D_{1\infty} - \xi^D_{1j}) + [\boldsymbol{\Upsilon}_2](\xi^D_{2\infty} - \xi^D_{2j}) + [\boldsymbol{\Upsilon}_3](\xi^D_{3\infty} - \xi^D_{3j}) + [\boldsymbol{\Upsilon}_4](\xi^D_{4\infty} - \xi^D_{4j})).
\end{aligned} \tag{A2}$$

It is evident that the last term in (39) vanishes as j tends to infinity and we obtain the desired weak $*$ lower semicontinuity:

$$\lim_{j\to\infty} \sum_{l=1}^{L} w_l W_l(v_l^{*j}) \geqslant \sum_{l=1}^{L} w_l W_l(v_l^{*\infty}). \tag{A3}$$

Existence of the optimal design now follows directly from the weak $*$ compactness of the set of controls by **D**.

We observe that for any chattering sequence of ply angle functions $\{\gamma^j(z)\}_{j=1}^{\infty}$ converging weak $*$ in $\mathscr{L}^\infty[0,\pi]$, that the fundamental theorem of Young measures (cf. Ball (1989)) guarantees the existence of a probability measure $\vartheta_z(\psi)$ (defined for almost all z) with support in $[0,\pi]$ for which:

$$\int_{-1/2}^{1/2} z^{0,1,2}\,[\cos 2\gamma^j(z), \cos 4\gamma^j(z), \sin 2\gamma^j(z), \sin 4\gamma^j(z)]\,\mathrm{d}z \to$$

$$\int_{-1/2}^{1/2} z^{0,1,2} \int_0^\pi [\cos 2\psi, \cos 4\psi, \sin 2\psi, \sin 4\psi]\,\mathrm{d}\vartheta_z(\psi)\,\mathrm{d}z. \tag{A4}$$

One sees that the curve

$$\int_0^{\pi} [\cos 2\psi, \cos 4\psi, \sin 2\psi, \sin 4\psi] \, d\vartheta_z(\psi) \tag{A5}$$

takes values in the set **H**. Observing that every point in **H** can be written as a sum of three extreme points, we write $\vartheta_z(\psi) = \rho_z(\psi)\,dz$ where

$$\rho_z(\psi) = \sum_{j=1}^{3} w_j(z)\partial(\psi - \varphi_j(z)). \tag{A6}$$

Here $\sum_{j=1}^{3} w_j(z) = 1, w_j \geqslant 0$, and $\varphi_j \in [0, \pi]$ are given by explicit formulas, see Lipton (1994b). Summarizing, we have:

$$\int_{-1/2}^{1/2} z^{0,1,2} [\cos 2\gamma^j(z), \cos 4\gamma^j(z), \sin 2\gamma^j(z), \sin 4\gamma^j(z)] \, dz \to \int_{-1/2}^{1/2} z^{0,1,2} \mathbf{P}_{[1,2,3,4]}(z) \, dz. \tag{A7}$$

In this way we see that the generalized laminate parameters correspond to chattering sequences of ply angle functions with Young's measure specified by a density supported on at most three ply angles.

Last we observe that the exchange of max and min in eqn (23) follows from the remarks in Section 4 together with the continuity of the functional with respect to weak $*\ \mathscr{L}^{\infty}$ convergence of the generalized design parameters and the continuity and coercivity of the functional in the displacement.